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Exact Integrability in Quantum Field Theory

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The treatment of exactly integrable systems in various branches of two-dimensional classical and quantum physics has recently been placed in a unified framework by the development of the quantum inverse method[1-4]. This method consolidates a broad range of developments in classical nonlinear wave (soliton) physics, statistical mechanics, and quantum field theory. The essential technique for analyzing exactly integrable quantum systems was invented by Bethe in 1931[5], who constructed the eigenstates of the isotropic Heisenberg spin chain. In various forms, Bethe's ansatz has been used over the past 50 years to study a variety of spin chains, two-dimensional lattice statistical models, many-body problems, and quantum field theories. Ironically, the study of integrable classical field systems is of much more recent origin, dating from the fundamental work of Gardner, Greene, Kruskal, and Miura in 1967[6], who discovered the inverse scattering method for reducing the Korteweg-deVries equation to a sequence of linear problems. It is the quantum mechanical extension of the inverse scattering method and its relationship to the methods associated with Bethe's ansatz which form the subject of this talk.

I'll begin by reviewing the basic idea of the classical inverse method. In this method one solves the initial value problem for a nonlinear equation of motion by considering an auxiliary linear eigenvalue problem of the form

$$\frac{\partial}{\partial x}\Psi(x,\zeta) = i Q_1(x,\zeta)\Psi(x,\zeta), \qquad (1)$$

For the most familiar cases (e.g. the nonlinear Schrödinger and Sine-Gordon equations), $Q_1(x,\zeta)$ is a 2×2 matrix which depends on the eigenvalue ζ and on a local field $\varphi(x)$. A particular solution to (1) must be specified by a boundary condition. For example, a matrix solution may be specified at a point x_0 by $\Psi(x_0,\zeta) = I$ = identity matrix. The nonlinear equation of motion for $\varphi(x,t)$ is related to the linear problem by allowing the eigenvalue equation (1) to depend parametrically on time in such a way that the time dependence of the solution Ψ is given by

$$\frac{\partial}{\partial t}\Psi = i Q_2 \Psi, \qquad (2)$$

where Q_2 is another 2×2 matrix which also depends on the eigenvalue ζ

and on the field $\varphi(x,\lambda)$. Eqns. (1) and (2) form a "Lax pair" $\partial_\mu \Psi = \lambda Q_\mu \Psi$. The nonlinear equation for φ emerges as a consistency condition from cross-differentiation of (1) and (2), i.e.

$\frac{\partial^2}{\partial x^\mu \partial x^\nu} = \frac{\partial^2}{\partial x^\nu \partial x^\mu}$, which gives

$$F_{\mu\nu} = \partial_\nu Q_\mu - \partial_\mu Q_\nu + i[Q_\mu, Q_\nu] = 0. \qquad (3)$$

By various choices for the matrices Q_μ , the equation $F_{\mu\nu} = 0$ reduces to certain nonlinear equations of motion for the field φ which have the property of complete integrability. The matrix eigenvalue equation (1) was first introduced by Zakharov and Shabat[7] for the nonlinear Schrödinger equation, and various nonlinear equations which could be constructed from the consistencey condition (3) were first systematically explored by Ablowitz et al.[8]. Here we will focus on the nonlinear Schrödinger equation

$$i\partial_0 \varphi = -\partial_1^2 \varphi + 2c|\varphi|^2 \varphi , \qquad (4)$$

which provides the simplest and most completely explored example of the quantum inverse method. To recover this equation from (3), we choose

$$Q_0 = \begin{pmatrix} \frac{k^2}{2} - c\varphi^*\varphi & \sqrt{c}\left(k\varphi + i\varphi_x\right) \\ -\sqrt{c}\left(k\varphi^* - i\varphi_x^*\right) & -\frac{k^2}{2} + c\varphi^*\varphi \end{pmatrix} \qquad (5a)$$

$$Q_1 = \begin{pmatrix} \frac{1}{2}k & \sqrt{c}\,\varphi \\ -\sqrt{c}\,\varphi^* & -\frac{1}{2}k \end{pmatrix} \qquad (5b)$$

Here and in the following discussion we will denote the eigenvalue ζ by $\hat{k}$ to emphasize its eventual role as a momentum variable in Bethe's ansatz.

The method for solving (4) consists of a sequence of steps which are quite analogous to the use of direct and inverse Fourier transformation to solve a linear equation of motion. At a fixed time, the linear problem is used to define a transformation from the field $\varphi(x)$ to a set of scattering data $a(k)$, $b(k)$ which are obtained from the asymptotic (in x) behavior of a solution to (1). This will be referred to as the direct transform. The inverse transform, which is accomplished by the Gel'fand-Levitan technique maps the scattering data back to the local field φ. Instead of dwelling on the classical method, we will proceed directly to the quantum nonlinear Schrödinger model (also known as the delta-function gas) for which the field φ is an operator with equal time commutation relations $[\varphi(x), \varphi^*(y)] = \delta(x-y)$. For this case we construct an operator transform by considering a normal ordered linear problem

$$\frac{\partial}{\partial x}\Psi(x,k) = i :Q_1(x,k)\,\Psi(x,k): \qquad (6)$$

where Q_1 is again given by (5b) and Ψ is a normal ordered operator functional of φ and φ^* which may be written explicitly as a path-ordered exponential string operator,

$$\Psi(x,k) = :P\exp i\int_{x_0}^{x} Q_1(y,k)\,dy: \qquad (7)$$

Note that the path ordering refers to the matrix structure, while the normal ordering refers to the operator structure.

In a unbounded system with a finite number of particles (zero density) we may assume that $\varphi(x) \to 0$ as $x \to \pm\infty$ in the sense of weak convergence. I'll comment later on how to recover results for finite density systems from this formalism. With this assumption we find that

$$\Psi(x,k) \xrightarrow[x\to\pm\infty]{} V(x,k) \times (\text{constant matrix}), \qquad (8)$$

where

$$V(x,k)=\begin{pmatrix} e^{ikx/2} & 0 \\ 0 & e^{-ikx/2} \end{pmatrix} . \qquad (9)$$

The scattering data operators $a(k)$ and $b(k)$ are defined by comparing the behavior of Ψ at $-\infty$ and $+\infty$, specifically

$$\mathcal{T}(k) = \lim_{\substack{x\to+\infty \\ x_0\to-\infty}} V^{-1}(x,k)\,\Psi(x,k)\,V(x_0,k)$$

$$\equiv \begin{pmatrix} a(k) & b^*(k) \\ b(k) & a^*(k) \end{pmatrix} . \qquad (10)$$

The central property of the a and b operators is that they obey simple commutation relations. These are most elegantly derived by the method of Sklyanin[1], which was pattered after Baxter's treatment of the eight vertex model . One compares equations for the quantities

$$H_{12}(x) = \Psi(x,k_1)\otimes\Psi(x,k_2) \qquad \text{and} \qquad H_{21}(x) = \Psi(x,k_2)\otimes\Psi(x,k_1) .$$

From (6), we see that

$$\frac{\partial}{\partial x} H_{12} = i\,{:}Q_1\Psi_1{:}\otimes\Psi_2 + i\,\Psi_1\otimes{:}Q_2\Psi_2{:} = i\,{:}\Gamma_{12}H_{12}{:} , \qquad (11)$$

where the subscripts refer to the eigenvalue, and

$$\Gamma_{12} = Q_1\otimes I + I\otimes Q_2 - ic\,\sigma^+\otimes\sigma^- . \qquad (12)$$

In Eq. (12), $\sigma^{\pm}$ are Pauli matrices, and the last term arises from normal ordering (11). Similarly, by interchanging eigenvalues, we get

$$\frac{\partial}{\partial x} H_{21} = i\,{:}\Gamma_{21}H_{21}{:} . \qquad (13)$$

The fundamental property of the matrices Γ_{12} and Γ_{21} is that they

$|k_1, \ldots, k_N\rangle$ are found to be precisely the same as the states constructed by Lieb and Liniger[10] in 1963 using Bethe's ansatz. This establishes the essential connection between quantum inverse methods and Bethe's ansatz. The operator $a(k)$ is the generating function for an infinite number of conserved operators. Another useful operator is the quantized reflection coefficient $R^*(k) = b(k)\, a^{-1}(k)$, which also creates eigenstates of H, but with a different normalization. The states created by R^* are in fact the properly normalized states. R and R^* have very simple commutation relations which are central to the formulation of the inverse (Gel'fand-Levitan) transform to be discussed next,

$$R^*(k_1)\, R^*(k_2) = e^{i\Delta(k_1 - k_2)}\, R^*(k_2) R^*(k_1)\,, \tag{20a}$$

$$R(k_1)\, R^*(k_2) = e^{-i\Delta(k_1 - k_2)}\, R^*(k_2) R(k_1) + 2\pi\delta(k_1 - k_2)\,, \tag{20b}$$

where $e^{i\Delta(k)} = (k - ic)/(k + ic)$ is the two-body phase shift.

It should be remarked that all of these results have been derived in an infinite volume. Since commutation relations of the form (17) hold not only for $\mathcal{T}$ but for the solution Φ itself, the commutator algebra of a's and b's in a finite box may also be obtained by the same method. This is the most straightforward route to discussion of finite density systems, but it leads to the following related complications : (1) Only $a(k) + a^*(k) = \mathrm{Tr}\, \mathcal{J}(k)$ can be diagonalized, and not $a(k)$ and $a^*(k)$ separately ; (2) the b-states (18) are eigenstates of $a + a^*$ only if the values of $k_1, \ldots, k_N$ satisfy certain constraints, which are precisely the periodic boundary conditions of Bethe's ansatz in a finite box ; (3) $R(k)$ is not a useful operator. All this makes the formulation of the Gel'fand-Levitan technique for recovering $Q(x)$ from the scattering data operators difficult if not impossible to carry out in a finite box. However, a suœssful alternative has been formulated which avoids the introduction of a finite box but recovers the correct finite density results by summing a fugacity expansion at finite temperature, as discussed below.

To formulate the Gel'fand-Levitan method[11], we consider a column vector Jost solution

are related by a C-number similarity transformation

$$\Gamma_{21} = \mathcal{R}\,\Gamma_{12}\,\mathcal{R}^{-1} \tag{14}$$

where

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & \alpha & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{15}$$

and

$$\alpha = \frac{k_1 - k_2}{k_1 - k_2 - i c}\,, \qquad \beta = \frac{-ic}{k_1 - k_2 - ic} \tag{16}$$

From this one finds that the quantities H_{12} and H_{21} are related by the same similarity transformation, which yields asymptotically a set of commutation relations for the scattering data operators a, a^*, b, and b^*,

$$\mathcal{R}\left[\mathcal{T}(k_1) \otimes \mathcal{T}(k_2)\right] = \left[\mathcal{T}(k_2) \otimes \mathcal{T}(k_1)\right]\mathcal{R}\,. \tag{17}$$

From the commutation relations (17), it may be shown that states created by the b-operators acting on the vacuum (zero-particle) state,

$$|k_1, k_2, \ldots, k_N\rangle \equiv b(k_1)\, b(k_2) \ldots b(k_N)\,|0\rangle \tag{18}$$

are eigenstates of the operators $a(k)$ for any value of k. Moreover, the Hamiltonian of the system $H = \int dx \left\{ \partial_1 \varphi^* \partial_1 \varphi + c\, \varphi^* \varphi^* \varphi \varphi \right\}$ may be constructed from the coefficients of the expansion of k in powers of $1/k$, and is therefore also diagonal on the states (18),

$$H\,|k_1, k_2, \ldots, k_N\rangle = \left(\sum_{i=1}^{N} k_i^2 \right) |k_1, k_2, \ldots, k_N\rangle. \tag{19}$$

$$\chi(x,k) = \begin{pmatrix} \chi_1(x,k) \\ \chi_2(x,k) \end{pmatrix} \tag{21}$$

of the Zaharov-Shabat problem $\partial_1 \chi = i k Q_1 \chi$, with boundary condition for real k,

$$\chi \xrightarrow[x\to\infty]{} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ikx/2}, \tag{22}$$

We also need the conjugate solution

$$\tilde{\chi} = \begin{pmatrix} \chi_2^* \\ \chi_1^* \end{pmatrix}. \tag{23}$$

The Gel'fand-Levitan integral equation follows from a dispersion relation for a (weakly) analytic operator function $\Phi(x,k)$, which is equal to $\tilde{\chi}\, e^{-ikx/2}$ in the upper half-plane and has a discontinuity of $i\sqrt{c}\, R^*(k)\, \chi(x,k)\, e^{-ikx/2}$ across the real axis. It can be shown that, for repulsive coupling, $c > 0$, Φ is also analytic in the lower half-plane and satisfies a simple dispersion relation, leading to a pair of coupled integral equations for χ_1 and χ_2^*,

$$\chi_2^*(x,k)\, e^{-ikx/2} = 1 + \frac{\sqrt{c}}{2\pi} \int dk' \, \frac{R^*(k')\, \chi_1(x,k')\, e^{-ik'x/2}}{k' - k - i\epsilon}, \tag{24a}$$

$$\chi_1(x,k)\, e^{ikx/2} = \frac{\sqrt{c}}{2\pi} \int dk' \, \frac{\chi_2^*(x,k')\, R(k')\, e^{ik'x/2}}{k' - k + i\epsilon}. \tag{24b}$$

These may be solved by iteration. The field operator $\varphi(x)$ is recovered from the Jost solution χ by letting $k \to \infty$ and noting the asymptotic behavior

$$\chi(x,k)\, e^{ikx/2} \xrightarrow[k\to\infty]{} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{\sqrt{c}}{ik} \begin{pmatrix} \varphi(x) \\ i\sqrt{c} \int_x^\infty \varphi^* \varphi \, dy \end{pmatrix} + O\left(\frac{1}{k^2}\right). \tag{25}$$

This gives $\varphi(x)$ and $j_0(x) = \varphi^*(x)\varphi(x)$ as functionals of $R(k)$ and $R^*(k)$:

$$\varphi(x) = \sum_{n=0}^{\infty} \varphi^{(n)}(x) , \tag{26}$$

$$j_0(x) = \sum_{n=0}^{\infty} j_0^{(n)}(x) , \tag{27}$$

where

$$\varphi^{(0)}(x) = \int \frac{dk}{2\pi} R(k) e^{ikx} = \tilde{R}(x) , \tag{26a}$$

$$\varphi^{(1)}(x) = -c \int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{dp_1}{2\pi} e^{i(k_1+k_2-p_1)x} \frac{R^*(p_1)R(k_1)R(k_2)}{(p_1-k_1-i\epsilon)(p_1-k_2-i\epsilon)} , \tag{26b}$$

$$\vdots$$

and

$$j_0^{(0)}(x) = \int \frac{dk_1}{2\pi}\frac{dp_1}{2\pi} (p_1-k_1) e^{-i(p_1-k_1)x} \frac{R^*(p_1)R(k_1)}{(p_1-k_1-i\epsilon)} , \tag{27a}$$

$$j_0^{(1)}(x) = \int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{dp_1}{2\pi}\frac{dp_2}{2\pi} (\Sigma p - \Sigma k) e^{-i(\Sigma p-\Sigma k)x} \frac{R^*(p_2)R^*(p_1)R(k_1)R(k_2)}{(p_1-k_1-i\epsilon)(p_1-k_2-i\epsilon)(p_2-k_2-i\epsilon)} \tag{27b}$$

The expansion (26) for $\varphi(x)$ has a very interesting property in the limit of infinitely repulsive coupling, $c \to \infty$ [12]. Surprisingly, in what looks like an expansion in positive powers of c, the $c \to \infty$ limit can be taken term by term after symmetrizing the integrands and using the RR and R^*R^* commutation relations (20a). For example, the second term $\varphi^{(1)}$ gives in this way

$$\varphi^{(1)}(x) = -c \int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{dp_1}{2\pi} e^{i(k_1+k_2-p_1)x} \frac{R^*(p_1)R(k_1)R(k_2)}{(p_1-k_1-i\epsilon)(p_1-k_2-i\epsilon)} \times \left(\frac{k_1-k_2}{k_1-k_2+ic}\right) , \tag{28}$$

which has the finite $c \to \infty$ limit

where $\tilde{R}(x)$ is the Fourier transform of the reflection coefficient,

$$B(x) = -2\int_x^\infty \tilde{R}^*(y)\,\tilde{R}(y)\,dy, \tag{30}$$

and N_R indicates normal ordering with respect to the R operators (i.e. R^*'s to the left and R's to the right). Term-by-term analysis of the series (26), using the same procedure of symmetrization of integrands, shows that

$$\varphi^{(n)}(x) = N_R\left\{\frac{1}{n!}\left[B(x)\right]^n \tilde{R}(x)\right\} \tag{31}$$

The Gel'fand-Levitan series for $\varphi(x)$ thus exponentiates into a Jordan-Wigner transformation in the limit $c \to \infty$,

$$\begin{aligned}\varphi(x) &= N_R\left\{\exp\left(-2\int_x^\infty \tilde{R}^*(y)\,\tilde{R}(y)\,dy\right)\tilde{R}(x)\right\} \\ &= \exp\left(\lambda\pi\int_x^\infty \tilde{R}^*(y)\,\tilde{R}(y)\,dy\right)\tilde{R}(x)\,.\end{aligned} \tag{32}$$

We note also that in the limit $c \to \infty$, the two-body phase shift $e^{\lambda\Delta} \to -1$, and the algebra of the R and R^* operators reduces to canonical fermion anticommutation relations, e.g. in coordinate space $\{\tilde{R}(x), \tilde{R}(y)\} = \{\tilde{R}^*(x), \tilde{R}^*(y)\} = 0$ and $\{\tilde{R}(x), \tilde{R}^*(y)\} = \delta(x-y)$. The Jordan-Wigner transformation is a standard trick in other contexts for converting boson operators to fermion operators. Here it arises as a special case of the Gel'fand-Levitan transformation. The nature of this transformation in other models, particularly the two-dimensional Ising model suggests a possible connection between exact integrability and inverse scattering transformations on one hand and Kramers-Wannier duality and order-disorder transformations on the other.

The Gel'fand-Levitan transformation for the charge density $j_0(x)$, Eq. (27), also has some interesting properties which provide the key to treatment of finite density systems without the introduction of a finite box[13]. We wish to calculate the partition function

$$Q = \mathrm{Tr}\, e^{-\beta(H-\mu N)}, \tag{33}$$

or more precisely, the connected part $\ln Q$. This may be related to certain "almost forward" matrix elements of the charge density $j_0(x)$,

$$\ln Q = \lim_{q \to 0} \mathrm{Tr}\, Y(q)\, e^{-\beta(H-\mu N)} , \tag{34}$$

where

$$Y(q) = e^{-iqK}\, N^{-1} \int_{-\infty}^{\infty} j_0(x)\, e^{iNqx}\, dx , \tag{35}$$

with K = Galilean boost operator. The operation of inserting $Y(q)$ inside the trace and taking the limit $q \to 0$ outside the trace simply picks out the connected part. Using the Gel'fand-Levitan expression (27) for $j_0(x)$ along with the algebra of the R operators, we obtain the thermodynamics of the system at finite density and temperature first derived from Bethe's ansatz by Yang and Yang[14]. The pressure P is given by

$$P = \int \frac{dk}{2\pi} \ln\left(1 + e^{-\beta \epsilon(k)}\right) , \tag{36}$$

where $\epsilon(k)$ is the solution to a nonlinear integral equation

$$\epsilon(k) = k^2 - \mu - \frac{1}{\beta} \int \frac{dk'}{2\pi} K(k-k') \ln\left(1 + e^{-\beta \epsilon(k')}\right) . \tag{37}$$

Here the kernel K is given by

$$K(k) = \frac{2c}{k^2 + c^2} \tag{38}$$

The expansion of the pressure in powers of the kernel K is in one-to-one correspondence with the Gel'fand-Levitan expansion of the charge density $j_0(x)$, Eq. (27). The zero temperature ($\beta \to \infty$) limit of (37) reduces to a linear integral equation

$$\epsilon(k) = k^2 - \mu + \int_{-k_F}^{k_F} \frac{dk'}{2\pi} K(k-k') \epsilon(k') . \tag{39}$$

where the "Fermi level" k_F is determined by $\epsilon(\pm k_F) = 0$. The spectrum of excitations above the ground state is given simply in terms of the function $\epsilon(k)$. (It consists of particles with energy $\epsilon(k)$ and holes with energy $-\epsilon(k)$.) This spectrum was first obtained by Lieb, using Bethe's ansatz and periodic boundary conditions in a box. Here, we have managed to obtain the fundamental spectral equation (37) or (39) without introducing a box and ruining the simple properties of the R operators. The use of the Gel'fand-Levitan formalism in finite temperature calculations of the type described here may offer a promising approach to the study of Green's functions for exactly integrable theories.

REFERENCES

1. E. Sklyanin and L.D. Faddeev, Dokl. Akad. Nauk. SSR 243, 1930 (1978) [Sov. Phys. Dokl. 23, 902 (1978)] ; E. Sklyanin, ibid., 244, 1337 (1979) [ibid., 24, 107 (1979)].
2. H.B. Thacker and D. Wilkinson, Phys. Rev. D19, 3660 (1979).
3. J. Honerkamp, P. Weber, and A. Wiesler, Nucl. Phys. B 152, 266 (1979).
4. For reviews see L. Faddeev, Steklov Math. Inst. Rep. P-2-79 ; H.B. Thacker, Fermilab Pub 80/38 -THY.
5. H. Bethe, Z. Physik 71, 205 (1931)
6. C.S. Gardner, J.M. Greene, M.D. Kruskal, and R.M. Miura, Phys. Rev. Lett. 19, 1095 (1967).
7. V.E. Zakharov and A.B. Shabat, Zh. Eksp. Teor. Fiz. 61, 118 (1971) [Soviet Phys. JETP 34, 62 (1972)].
8. M. Ablowitz, D.J. Kaup, A.C. Newell, and H. Segur, Phys. Rev. Lett. 30, 1262; 31, 125 (1973)
9. R.J. Baxter, Ann. Phys. 70, 193 (1972).
10. E.H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963) ; E.H. Lieb, ibid., 130, 1616 (1963).
11. D.B. Creamer, H.B. Thacker, and D. Wilkinson, Phys. Rev. D 21, 1523 (1980)
12. D.B. Creamer, H.B. Thacker, and D. Wilkinson, Phys. Lett. B
13. D.B. Creamer, H.B. Thacker, and D. Wilkinson, Fermilab Pub 80/25-THY.
14. C.N. Yang and C.P. Yang, J. Math. Phys. 10, 1115 (1969).
